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RESUMMED COEFFICIENT FUNCTION FOR THE SHAPE FUNCTION

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Abstract

We present a leading evaluation of the resummed coefficient function for the shape function. It is also shown that the coefficient function is short-distance-dominated. Our results allow relating the shape function computed on the lattice to the physical QCD distributions.

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1 Introduction

In this note we present a leading evaluation of the coefficient function of the shape function [1]–[3]. The latter is also called structure function of the heavy flavours. The coefficient function allows relating the shape function — computed with a non-perturbative technique such as lattice QCD — to distributions in semi-inclusive heavy-flavour decays. We consider in particular the processes

$$B \to X_s + \gamma \tag{1}$$

and

$$B \to X_u + l + \nu \tag{2}$$

in the hard limit

$$Q \gg \Lambda. \tag{3}$$

The quantity Q is the hard scale of these time-like processes: $Q \equiv 2\mathcal{E}$, where $\mathcal{E}$ is the hadronic energy of the state X and Λ is the QCD scale[1]. For a B-meson at rest, $v = (1; 0, 0, 0)$, and the jet X flying along the minus direction ($-z$ axis), the shape function is defined as

$$\varphi\left(k_+\right) \equiv \langle B\left(v\right)| h_v^\dagger \delta\left(k_+ - iD_+\right) h_v | B\left(v\right)\rangle. \tag{4}$$

This represents the probability that the b-quark in the B-meson has momenta

$$p_b = m_B v + k' \tag{5}$$

with any transverse and minus components and with given plus component

$$k'_+ = k_+. \tag{6}$$

The static field $h_v\left(x\right)$ is related to the Dirac field of the beauty quark $b\left(x\right)$ by

$$b\left(x\right) = e^{-i m_B v\cdot x} h_v\left(x\right) + O\left(\frac{1}{m_B}\right). \tag{7}$$

With the shape function, the decays (1) and (2) are related to their respective quark-level processes

$$b \to \widehat{X}_s + \gamma \tag{8}$$

and

$$b \to \widehat{X}_u + l + \nu, \tag{9}$$

where the b-quark has the momentum (5) with the distribution (4). It can be shown that the invariant mass m of the state $\widehat{X}$ is related to the virtuality of the b-quark by

$$k_+ = -\frac{m^2}{Q}. \tag{10}$$

The shape function is a non-perturbative distribution — analogous to parton distribution functions — and it describes the slice of the semi-inclusive region[2] in which

$$m^2 \sim \Lambda\, Q,$$

[1]For the rare decay (1), one can actually set $Q = m_B$.

[2]This region is also called threshold region, large-x region, radiation-inhibited region and Sudakov region.

i.e.

$$|k_+| \sim \Lambda, \qquad \text{or} \qquad z \sim 1 - \frac{\Lambda}{Q}, \tag{11}$$

where

$$z \equiv 1 - \frac{m^2}{Q^2}. \tag{12}$$

Because of ultraviolet divergences affecting its matrix elements, the shape function has a dependence on the ultraviolet cut-off or renormalization point μ :

$$\varphi(k_+) = \varphi(k_+;\mu), \tag{13}$$

and is related to a physical QCD distribution by means of a coefficient function by (cf. eq. (77))

$$\boldsymbol{\varphi}(k_+;Q) = \int dk'_+ \, C\left(k_+ - k'_+; Q, \mu\right) \, \varphi\left(k'_+;\mu\right). \tag{14}$$

The QCD distribution does not depend on μ :

$$\frac{d}{d\mu}\boldsymbol{\varphi}(k_+;Q) = 0, \tag{15}$$

while the shape function does not depend on Q. As a consequence, the coefficient function depends on both Q and μ.

The shape function can be computed with a non-perturbative technique or extracted from experimental data. If it is computed inside a field-theory model — such as lattice QCD — its expression will exhibit a μ-dependence that cancels against that of the coefficient function. If it is instead computed inside a phenomenological model — such as a quark model — the situation is less transparent. The μ-independence is not "automatic" and one has to specify the value of μ appropriate for the model. Some care is needed also in extracting the shape function from the experimental data, in order to avoid double counting of perturbative corrections. A factorization scheme must be defined and the coefficient functions for the various processes all have to be computed in the same scheme[3]. In particular, if a branching MonteCarlo is used for the analysis, the perturbative corrections generated by the program must be subtracted.

The coefficient function is obtained by evaluating in leading approximation both $\boldsymbol{\varphi}$ and φ and inserting their expression in eq. (14). Since the coefficient function is expected to be a short-distance quantity, we compute the QCD distribution and the shape function in PT for an on-shell b-quark ($k' = 0$). This expectation will be verified *a posteriori.*

2 The QCD distribution

The (perturbative) long-distance effects occurring in (1) and (2) can be factorized in the function

$$\mathbf{f}(z) = \delta(1 - z - 0) - A_1 \alpha_S \left(\frac{\log[1-z]}{1-z}\right)_+, \tag{16}$$

where

$$A_1 = \frac{C_F}{\pi} \tag{17}$$

[3]The situation is analogous to usual hard processes, where various factorization schemes for the parton distribution functions are defined: DIS, $\overline{\text{MS}}$, etc.

and $C_F = \left(N_c^2 - 1\right) / \left(2N_c\right) = 4/3$. The plus-distribution is defined as usual as

$$P(z)_+ \equiv P(z) \, - \, \delta(1 - z - 0) \int_0^1 dy \, P(y) \,. \tag{18}$$

The integrated or cumulative distribution is defined as

$$\mathbf{F}(z) \equiv \int_z^1 dz' \, \mathbf{f}(z') \,. \tag{19}$$

Inserting expression (16) in this, one obtains the well-known double logarithm:

$$\mathbf{F}(z) = 1 - \frac{A_1 \alpha_S}{2} \log^2(1 - z) \,. \tag{20}$$

The cumulative distribution satisfies the normalization condition $\mathbf{F}(0) = 1$. Multiple soft-gluon emission exponentiates the one-loop distribution, so that

$$\mathbf{F}(z) = \exp\left[-\frac{A_1 \alpha_S}{2} \log^2(1 - z) \right] . \tag{21}$$

For further improvement, it is convenient to write the function $\mathbf{f}(z)$ in an "unintegrated" form, as

$$\mathbf{f}(z) = \delta(1 - z) + A_1 \alpha_S \int_0^1 \frac{d\epsilon}{\epsilon} \int_0^1 \frac{dt}{t} \left[\delta(1 - z - \epsilon t) - \delta(1 - z) \right] , \tag{22}$$

where we have defined the unitary energy and angular variables

$$\epsilon \equiv \frac{E}{Q} \qquad \text{and} \qquad t \equiv \frac{1 - \cos\theta}{2} . \tag{23}$$

The quantity E is two times the energy of the soft gluon, $E = 2E_g$, and θ is the emission angle. Leading logarithmic corrections are included replacing the bare coupling with the running coupling evaluated at the gluon transverse momentum squared [4]:

$$\alpha_S \to \alpha_S\left(l_\perp^2\right) , \tag{24}$$

with

$$l_\perp^2 \simeq E_g^2 \, \theta^2 \simeq Q^2 \epsilon^2 t . \tag{25}$$

The QCD semi-inclusive form factor then reads:

$$\mathbf{f}(z) = \delta(1 - z) + \int_0^1 \frac{d\epsilon}{\epsilon} \int_0^1 \frac{dt}{t} A_1 \alpha_S\left(Q^2 \epsilon^2 t\right) \left[\delta(1 - z - \epsilon t) - \delta(1 - z) \right] . \tag{26}$$

Performing the integrations, one obtains:

$$\mathbf{f}(z) = \delta(1 - z) + \frac{A_1}{\beta_0} \left\{ \left[\frac{\log\log\xi(1 - z)}{1 - z} \right]_+ - \left[\frac{\log\log\xi(1 - z)^2}{1 - z} \right]_+ \right\} , \tag{27}$$

where $\beta_0 \equiv (11/3 \, N_c - 2/3 \, n_f) / (4\pi)$ and ξ is the square of the ratio of the hard scale to the QCD scale:

$$\xi \equiv \frac{Q^2}{\Lambda^2} . \tag{28}$$

Equation (27) replaces the frozen-coupling result (16) and reduces to it in the limit $\xi \to \infty$ with z fixed. The integrated distribution reads

$$\mathbf{F}(z) = 1 - \frac{A_1}{2\beta_0} \left[\log\xi \log\log\xi - 2\log\xi(1 - z) \log\log\xi(1 - z) + \log\xi(1 - z)^2 \log\log\xi(1 - z)^2 \right] . \tag{29}$$

As in the frozen-coupling case, higher-order corrections exponentiate the one-loop result, so that

$$\mathbf{F}(z) = \exp\left\{ -\frac{A_1}{2\beta_0} \left[\log\xi \log\log\xi - 2\log\xi\,(1-z)\log\log\xi\,(1-z) + \log\xi\,(1-z)^2 \log\log\xi\,(1-z)^2 \right] \right\}. \quad (30)$$

Equation (30) replaces the frozen-coupling result (21).

The above distributions are "physical" and therefore should be real for any value of z in the range

$$0 \le z \le 1. \quad (31)$$

In practice, the distributions (27) and (30) are real only if the range of z is restricted to

$$z < 1 - \frac{\Lambda}{Q} \qquad \text{or} \qquad m^2 > \Lambda\, Q. \quad (32)$$

This restriction is absent in the frozen coupling case and originates from the infrared pole in the running coupling, which diverges when the transverse gluon momentum becomes as small as the QCD scale. The resummed distribution is therefore reliable as long as one stays away from the infrared pole, which implies the limitation

$$m^2 \gg \Lambda\, Q. \quad (33)$$

This condition makes the kinematic region (11) unaccessible to the QCD distribution and forces the introduction of a non-perturbative component, namely the shape function. The physical origin of the restriction (33) is easily understood with the following qualitative considerations. The jet mass and the gluon transverse momentum are given by

$$\begin{aligned} \frac{m^2}{Q} &\sim E\,\theta^2, \\ l_\perp &\sim E\,\theta. \end{aligned} \quad (34)$$

The smallest transverse momentum for fixed invariant mass is obtained for $\theta \sim 1$ and is

$$l_{\perp \min} \sim \frac{m^2}{Q}. \quad (35)$$

In region (11) this momentum is of the order of the QCD scale, where the coupling leaves the perturbative phase. The conclusion is that large-angle soft-gluon emission signals non-perturbative effects in the region (11).

In general, resummation of multiple soft-gluon emission is performed in Mellin space. We therefore consider the moments of the form factor:

$$\begin{aligned} \mathbf{f}_N &\equiv \int_0^1 dz\, z^N\, \mathbf{f}(z) \\ &= 1 + \Delta\mathbf{f}_N. \end{aligned} \quad (36)$$

Exponentiation of the "effective"[4] one-gluon distribution takes place, so that

$$\mathbf{f}_N = e^{\Delta \mathbf{f}_N}. \quad (37)$$

Using the large-N approximation [5]

$$(1-y)^N - 1 \simeq -\theta\,(y - 1/n)\,, \quad (38)$$

[4]We call it "effective" because the insertion of the running coupling in the time-like region already includes some multiple gluon-emission effects.

where $n \equiv N/N_0$ with $N_0 \equiv e^{-\gamma_E} \simeq 0.56$ and $\gamma_E \simeq 0.58$ is the Euler constant, the moments of eq. (16) or (22) read (frozen-coupling case, $\beta_0 \to 0$)

$$\mathbf{f}_n = \exp\left[-\frac{A_1 \alpha_S}{2} \log^2 n\right]. \tag{39}$$

In the running coupling case, the Mellin transform of eq. (26) or (27) is [6, 7, 5]:

$$\mathbf{f}_n = \exp\left\{-\frac{A_1}{2\beta_0}\left[\log\frac{\xi}{n^2}\log\log\frac{\xi}{n^2} - 2\log\frac{\xi}{n}\log\log\frac{\xi}{n} + \log\xi\log\log\xi\right]\right\}. \tag{40}$$

Note that $\mathbf{f}_n = 1$ for $n = 1$. The distribution (40) is usually written as

$$\mathbf{f}_n = \exp\left[l\ g_1\left(\beta_0 \alpha_S l\right)\right], \tag{41}$$

where $l \equiv \log n$, $\alpha_S \equiv \alpha_S(Q)$ and

$$g_1(w) \equiv -\frac{A_1}{2\beta_0}\frac{1}{w}\left[(1-2w)\log(1-2w) - 2(1-w)\log(1-w)\right]. \tag{42}$$

Expanding this function to lowest order in w, one obtains $g_1(w) = -A_1 w/(2\beta_0) + O\left(w^2\right)$ and recovers the fixed coupling result.

As we clearly see, the inverse transform from $\mathbf{f}_n$ to $\mathbf{F}(z)$ is simply computed with the replacement [8]

$$n \to \frac{1}{1-z}. \tag{43}$$

The limitation to the resummed perturbative result found before (eq. (33)) reads, in N-space:

$$n \ll \frac{Q}{\Lambda}. \tag{44}$$

3 The shape function

Let us now consider the quantity in the effective theory related to $\mathbf{f}(z)$, namely the shape function $\varphi(k_+)$. For an on-shell b-quark, the latter is given by

$$\varphi(k_+) = \delta(k_+) + A_1\alpha_S \frac{\theta(0; k_+; -\mu)}{-k_+}\log\frac{\mu}{-k_+} - A_1\alpha_S\,\delta(k_+)\int_{-\mu}^{0}\frac{dl_+}{-l_+}\log\frac{\mu}{-l_+}, \tag{45}$$

where $\mu \equiv 2\Lambda_S$ and we defined $\theta(a_1; a_2; \cdots a_n) \equiv \theta(a_1 - a_2)\,\theta(a_2 - a_3)\cdots\theta(a_{n-1} - a_n)$. The regularization used [3] imposes a cutoff on the spatial loop momenta and not on the energies:

$$|\overrightarrow{l}| < \Lambda_S, \qquad -\infty < l_0 < +\infty; \tag{46}$$

it is qualitatively similar to lattice regularization. Proceeding in a similar way, one can write

$$\varphi(k_+) = \delta(k_+) + \int_0^{\mu}\frac{dE}{E}\int_0^1\frac{dt}{t}A_1\alpha_S\left(E^2 t\right)\left[\delta(k_+ + Et) - \delta(k_+)\right]. \tag{47}$$

We explicitly see that the hard scale Q does not appear in $f(k_+)$, as it should. Since $k_+ = -Q(1-z)$, the shape function in the "QCD variable" z reads

$$f(z) = \delta(1-z) + \int_0^{\eta}\frac{d\epsilon}{\epsilon}\int_0^1\frac{dt}{t}A_1\alpha_S\left(Q^2\epsilon^2 t\right)\left[\delta(1-z-\epsilon t) - \delta(1-z)\right], \tag{48}$$

where we have defined an adimensional shape function as

$$f\,(z) \equiv Q\,\varphi\,(k_+)\,. \tag{49}$$

The quantity η is the ratio of the UV cutoff of the effective theory to the hard scale,

$$\eta \equiv \frac{\mu}{Q} < 1, \tag{50}$$

because the shape function is defined in a low-energy effective theory. To avoid substantial finite cut-off effects, it must also be assumed that

$$\mu \gg \Lambda. \tag{51}$$

Taking the Mellin moments and exponentiating the one-loop distribution as in the QCD case, one obtains[5]

$$f_N = e^{\Delta f_N}. \tag{52}$$

A straightforward computation gives

$$\log f_n = -\frac{A_1}{2\beta_0}\theta\,(n-1/\eta)\left[\log\frac{\xi}{n^2}\log\log\frac{\xi}{n^2} - 2\log\frac{\eta\xi}{n}\log\log\frac{\eta\xi}{n} + \log\eta^2\xi\log\log\eta^2\xi\right]. \tag{53}$$

As in the case of the QCD distribution, $f_n = 1$ for $n = 1$. The N-moments of the shape function are reliably computed in PT if one restricts n as in eq. (44). We therefore have here the same limitation of the QCD distribution, as expected on physical ground.

4 The coefficient function

We now introduce the coefficient function $C\,(z)$, relating the shape function to the QCD semi-inclusive form factor, as

$$C\,(z) \equiv \delta\,(1-z) + \Delta C\,(z)\,, \tag{54}$$

where

$$\Delta C\,(z) \equiv \mathbf{f}\,(z) - f\,(z)\,. \tag{55}$$

Inserting the expressions for the two distributions, we obtain

$$\Delta C\,(z) = \int_\eta^1 \frac{d\epsilon}{\epsilon}\int_0^1 \frac{dt}{t} A_1 \alpha_S\left(Q^2\epsilon^2 t\right)\left[\delta\,(1-z-\epsilon t) - \delta\,(1-z)\right]. \tag{56}$$

The expression for the coefficient function is similar to the one for the QCD form factor, the only difference being a lower cut-off on the gluon energies in this case. Note that there is no angular cutoff. The QCD distribution and the shape function are related by a convolution in momentum space,

$$\mathbf{f}\,(z) = \int_0^1\int_0^1 dz' dz'' \delta\,(z - z'z'')\; C\,(z')\; f\,(z'')\,. \tag{57}$$

Taking the moments on both sides, one diagonalizes the convolution, so that

$$\mathbf{f}_n = C_n\, f_n. \tag{58}$$

[5]Do not confuse *these* moments of the shape function f_N with the moments considered by other authors, $\int dk_+\, k_+^N\, \varphi\,(k_+) \sim \int dz\,(1-z)^N\, f\,(z)\,.$

The usual exponentiation reads:

$$C_n = e^{\Delta C_n}. \tag{59}$$

The coefficient function can be computed from the difference in eq. (55) or directly from the integral representation in eq. (56) in the following way. The integration over the emission angle t gives

$$\Delta C_N = \int_0^1 \frac{dy}{y}\left[(1-y)^N - 1\right]\int_{\max(\eta,\, y)}^1 \frac{d\epsilon}{\epsilon} A_1 \alpha_S\left(Q^2 \epsilon y\right), \tag{60}$$

where $y \equiv 1 - z$. The integration over y is done using the previous approximation (38) and one finally obtains:

$$\begin{aligned}\log C_n \;=\; & -\frac{A_1}{2\beta_0}\left\{\theta\left(1/\eta - n\right)\left[\log\frac{\xi}{n^2}\log\log\frac{\xi}{n^2} - 2\log\frac{\xi}{n}\log\log\frac{\xi}{n} + \log\xi\log\log\xi\right] + \right. \\ & \left. +\theta\left(n - 1/\eta\right)\left[2\log\frac{\eta\xi}{n}\log\log\frac{\eta\xi}{n} - 2\log\frac{\xi}{n}\log\log\frac{\xi}{n} + \log\xi\log\log\xi - \log\eta^2\xi\log\log\eta^2\xi\right]\right\}.\end{aligned} \tag{61}$$

The QCD function $\mathbf{f}_n$ is equal to the coefficient of $\theta\left(1/\eta - n\right)$ in the r.h.s of eq. (61). Collecting the terms in common to the two θ-functions, eq. (61) can also be written as:

$$\begin{aligned}\log C_n \;=\; & -\frac{A_1}{2\beta_0}\left\{\theta\left(1/\eta - n\right)\log\frac{\xi}{n^2}\log\log\frac{\xi}{n^2} + \log\xi\log\log\xi - 2\log\frac{\xi}{n}\log\log\frac{\xi}{n} + \right. \\ & \left. +\theta\left(n - 1/\eta\right)\left[2\log\frac{\eta\xi}{n}\log\log\frac{\eta\xi}{n} - \log\eta^2\xi\log\log\eta^2\xi\right]\right\}.\end{aligned} \tag{62}$$

Let us comment on the result, represented by eq. (61) or by eq. (62). It is immediate to check that $C_n = 1$ when $\eta = 1$ or when $n = 1$, as it should. Furthermore, $C_n \to 1$ in the limit $Q \to \infty$ with n and μ fixed.

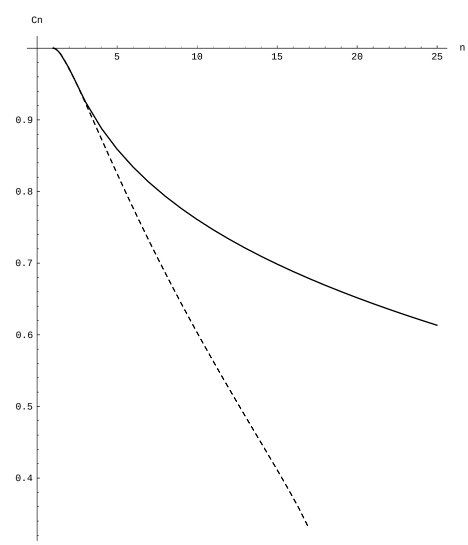

Figure 1: Plot of the resummed coefficient function C_n (solid line) and the QCD semi-inclusive form factor $\mathbf{f}_n$ (dashed line) for the choice of the parameters discussed in the text.

The main point is that the coefficient function is perturbatively computable as long as

$$n \ll \frac{Q\,\mu}{\Lambda^2}.$$

Note that this critical value of n is much larger than the one for the QCD form factor or the shape function given in eq. (44), because of eq. (51). The equivalent limitation in momentum space for $C\,(z)$ is

$$m^2 \gg \frac{Q\,\Lambda^2}{\mu}. \tag{63}$$

The occurrence of non-perturbative effects outside the region indicated by eq. (63) can be understood in a simple way. The problem is to find the smallest gluon transverse momentum for a given invariant mass, eqs. (34), with the additional constraint $E \gtrsim \mu$. The minimum occurs for the smallest gluon energy, $E \sim \mu$, where one has

$$\theta \sim \sqrt{\frac{m^2}{\mu\,Q}} \sim \sqrt{\frac{\Lambda}{\mu}} \ll 1 \tag{64}$$

and

$$l_{\perp\min} \sim \sqrt{\frac{m^2}{Q}\mu} \sim \sqrt{\Lambda\,\mu} \gg \Lambda, \tag{65}$$

in which conditions (11) and (51) have been used. A comparison between eqs. (35) and (65) shows that transverse momenta are substantially larger in the coefficient function than in the QCD distribution; this allows a perturbative treatment of the former. This is because the infrared cut-off on the gluon energies in $C\,(z)$ has the effect of lowering the largest emission angle, indirectly increasing the minimum gluon transverse momentum.

In single logarithmic problems, the loop expressions for the coefficient functions usually have phase-space restrictions of the form

$$l_\perp \gtrsim \mu \gg \Lambda. \tag{66}$$

Therefore, factorization with the shape function involves, as the relevant dynamical scale, a softer scale with respect to single logarithmic problems, because

$$\mu \gg \sqrt{\Lambda\,\mu}. \tag{67}$$

Non-perturbative corrections to the factorized form (57) or (58) are expected to be of the size

$$\frac{\Lambda^2}{l^2_{\perp\min}}, \tag{68}$$

which means of order

$$\frac{\Lambda^2}{|k_+|\,\mu} \sim \frac{\Lambda}{\mu}. \tag{69}$$

One inverse power of the factorization scale μ is involved. The corrections are therefore much larger than in single logarithmic problems, where the estimate (68) translates to

$$\frac{\Lambda^2}{\mu^2}, \tag{70}$$

involving two inverse powers of the factorization scale.

The previous findings can be summarized by considering eq. (58) in the various regions of the moment index n, corresponding to different dynamical regimes:

1. In the region[6]

$$1 \ll n \ll \frac{Q}{\Lambda}$$

or equivalently

$$\Lambda\, Q \, \ll \, m^2 \, \ll \, Q^2, \tag{71}$$

the QCD function, the shape function and the coefficient function are all reliably computed in perturbation theory. Actually, in this case there is no need to introduce the shape function and the resummed QCD form factor is all is needed: its splitting in a shape function and a coefficient function is irrelevant;

2. In the "more exclusive" region

$$\frac{Q}{\Lambda} \sim n \ll \frac{Q\, \mu}{\Lambda^2}$$

or equivalently

$$\Lambda\, Q \, \sim \, m^2 \, \gg \, \frac{Q\, \Lambda^2}{\mu}, \tag{72}$$

the QCD function and the shape function become non-perturbative, while the coefficient function is still perturbatively computable. The idea is that one replaces the perturbative evaluation of the shape function with a non-perturbative one and *defines* the QCD distribution by means of relation (58);

3. If one considers larger moments or, equivalently, smaller jet masses,

$$n \gtrsim \frac{Q\, \mu}{\Lambda^2} \qquad \text{or} \qquad m^2 \lesssim \frac{Q\, \Lambda^2}{\mu}, \tag{73}$$

also the coefficient function becomes non-perturbative and the shape function loses most of its meaning.

The coefficient function C_n is plotted in fig. 1 together with the QCD function $\mathbf{f}_n$. For small values of n the two curves are very close to each other because typical transverse momenta are large and the lower cut-off μ in C_n is ineffective. In the figure we have taken $Q = 5.2\,\mathrm{GeV}$, $\alpha_S = 0.24$, $\mu = 2\,\mathrm{GeV}$, $\Lambda = 0.3\,\mathrm{GeV}$ and $n_f = 3$. For these value of the parameters, $\mathbf{f}_n$ becomes singular for $n > 17$, while C_n becomes singular for $n > 115$. The coefficient function monotonically decreases with n, starting from $C_n = 1$ at $n = 1$ and going down to $C_n \simeq 0.68$ for $n = 17$. This decay is produced by gluons of energy between the cutoff μ and the hard scale Q. The function $\mathbf{f}_n$ also monotonically decreases with n starting again from $\mathbf{f}_n = 1$ at $n = 1$ and reaching $\mathbf{f}_n \simeq 0.33$ for $n = 17$. This faster decay is produced by gluons with any energy between the soft scale $|k_+|$ and Q.

The result (61) applies also to the coefficient function of the shape function defined in lattice regularization [9], after the identification is done

$$\frac{1}{a} = c\, \mu, \tag{74}$$

where c is a constant of order 1 and a is the lattice spacing. The precise value of c is determined only in next-to-leading approximation. One has to perform a full order α_S computation of the shape function in lattice regularization and equate it to the expansion to order α_S of the next-to-leading analogue of eq. (47).

[6]The region $n \sim 1$ is trivial as there are no large infrared logarithms and fixed order perturbation theory is sufficient.

5 Conclusions

In general, the QCD semi-inclusive form factor $\mathbf{f}(z)$ factorizes and resums large infrared double-logarithms which occur in the threshold region

$$m^2 \ll Q^2. \tag{75}$$

These terms are related to the emission of soft gluons with transverse momenta ranging from the infrared scale $|k_+| = m^2/Q$ up to the hard scale Q. As long as one keeps $|k_+| \gg \Lambda$, dynamics is controlled by perturbation theory. A phenomenologically relevant region is, however, $|k_+| \sim \Lambda$, in which substantial non-perturbative effects are encountered. We factorize them by means of the shape function. An additional, unphysical, scale μ is introduced in the problem, intermediate between the hard and the soft scale:

$$|k_+| \ll \mu \ll Q. \tag{76}$$

This means a splitting of the double logarithm in the QCD cumulative distribution of the form

$$\begin{aligned}\mathbf{F} &= 1 - \frac{A_1\alpha_S}{2}\log^2\frac{Q^2}{m^2} = 1 - \frac{A_1\alpha_S}{2}\log^2\frac{Q}{|k_+|} \\ &= \left(1 - \frac{A_1\alpha_S}{2}\log^2\frac{Q}{\mu} - A_1\alpha_S\log\frac{Q}{\mu}\log\frac{\mu}{|k_+|}\right)\left(1 - \frac{A_1\alpha_S}{2}\log^2\frac{\mu}{|k_+|}\right).\end{aligned} \tag{77}$$

The scale μ has the role of an IR cut-off for the coefficient function and of an UV cut-off for the shape function. The first parenthesis in (77) contains the coefficient function, which is short-distance-dominated and reliably computed in PT, while the second parenthesis contains a perturbative evaluation of the shape function. The latter computation is "unreliable", so it is thrown away and replaced by a non-perturbative one. The factorization scale μ then separates what we compute in PT (above μ) from what we give up to compute in PT (below μ). We may say that the coefficient function of the shape function is the "upper part" of a semi-inclusive QCD form factor.

We have shown that the transverse gluon momenta entering the coefficient function have a lower bound given by[7]

$$l_\perp \gtrsim \sqrt{|k_+|\,\mu} \sim \sqrt{\Lambda\,\mu} \gg \Lambda, \tag{78}$$

where eq. (11) has been used. As a consequence, non-perturbative corrections to the factorized form (57) or (58) are expected to be of the size

$$\frac{\Lambda}{\mu}, \tag{79}$$

involving one inverse power of the factorization scale μ. This situation is to be contrasted with single logarithmic problems, where corrections to factorization have typically a size

$$\frac{\Lambda^2}{\mu^2}, \tag{80}$$

involving two inverse powers of the factorization scale. We may say that factorization in this double-logarithmic problem is still consistent, but it is presumably "less lucky" than in single-logarithmic cases.

We conjecture that a similar factorization — into a perturbatively calculable coefficient function and a non-perturbative component — can be done for other semi-inclusive distributions, such as shape variables for very

[7] The appearance of a single logarithm of the soft scale $|k_+|$ in the coefficient function, eq. (77), should not be erroneously interpreted as the signal of a non-perturbative effect in C.

small values of the resolution parameters. An example is the thrust distribution in e^+e^- annihilations [10] for values of the jet masses in the region (11), for which

$$1 - T \sim \frac{\Lambda}{Q}. \tag{81}$$

In other words, we believe that the idea of the shape function can be generalized to describe many semi-inclusive distributions in the region (11).

To conclude, we have presented a leading evalutation of the resummed coefficient function of the shape function and we have proved that the latter is short-distance-dominated.

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